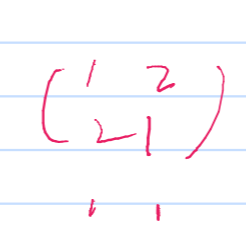
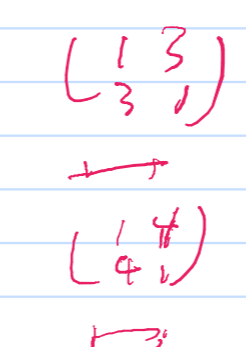


Chapter 1 Coxeter Group

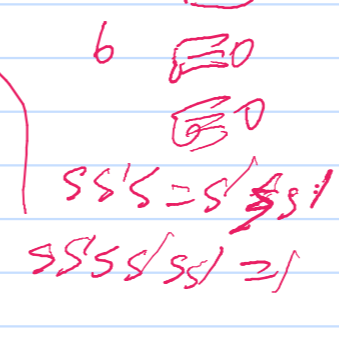
Defn 1.1 Let  $S$  be a finite set and let  $(m_{s,s'})_{(s,s') \in S \times S}$  be a matrix with entries in  $\mathbb{N} \cup \{\infty\}$ , s.t.  $m_{s,s'} = 1$  for all  $s$  and  $m_{s,s'} = m_{s',s} \geq 2$ .  $(m_{s,s'})$  is called a Coxeter matrix.



Defn 1.2 In the case, where  $m_{s,s'} \in \{2, 3, 4, 6, \infty\}$  for all  $s \neq s'$ , the matrix  $(m_{s,s'})_{(s,s') \in S \times S}$  is completely described by a graph (the Coxeter graph) with set of vertices in bijection with  $S$  where the vertices corresp to  $s \neq s'$  are joined by an edge, if  $m_{s,s'} = 3$ , double edge, if  $m_{s,s'} = 4$ , triple edge, if  $m_{s,s'} = 6$ , quadruple edge, if  $m_{s,s'} = \infty$ .



Defn 1.3 Let  $W$  be a group generated by  $S$ ,  $(ss')^{m_{s,s'}} = 1$ ,  $m_{s,s'} < \infty$ ,  $s^2 = 1$ .  $W$  is called a Coxeter group.

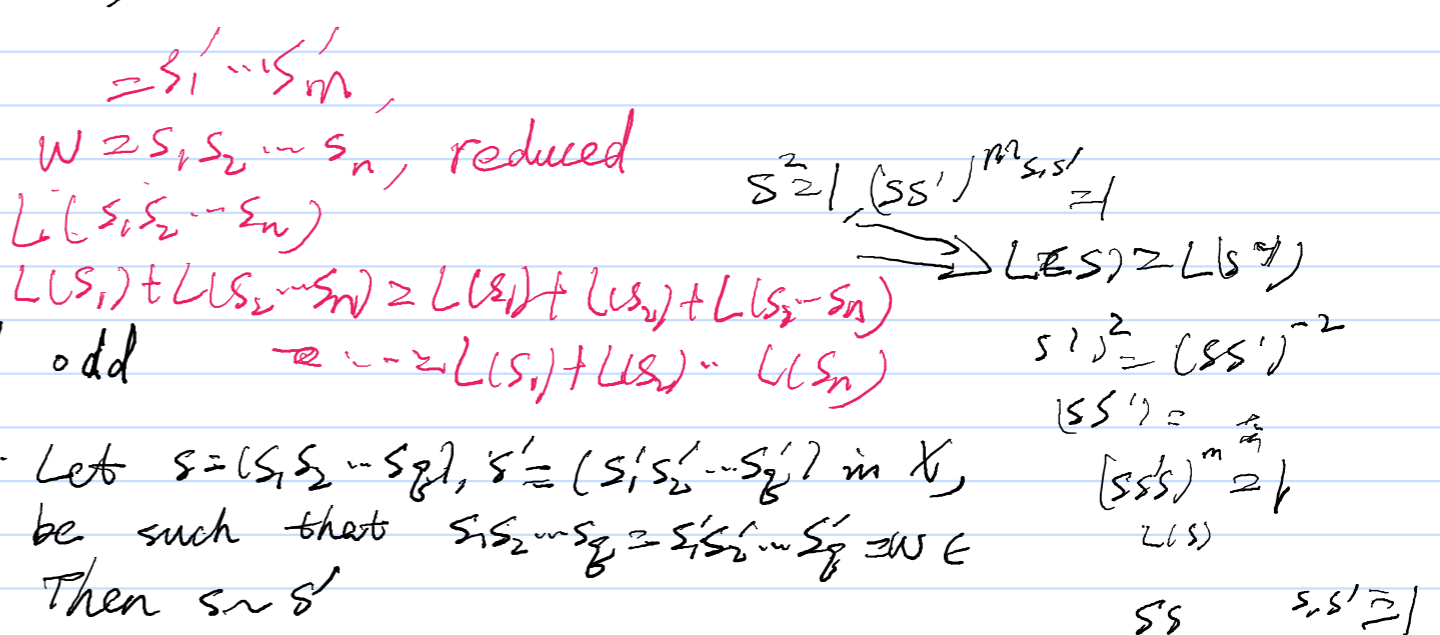


Chap 3. The Algebra  $H$

3.1. Let  $W, S$  be a Coxeter group. A map  $L: W \rightarrow \mathbb{Z}$  is said to be a weight function for  $W$  if  $L(ws) = L(w) + L(s)$ ,  $\forall w, s \in W$ , s.t.  $L(w) = L(w) + L(s)$ .

Defn: Assume that a weight function  $L: W \rightarrow \mathbb{Z}$  is fixed,  $\dots W, L$  is a weighted Coxeter group.

For ex:  $\dots L = l$ , split case. Note:  $L$  is determined by its value on  $S$  which are subject only to the condition.



?  $L(s) = L(s')$   $\forall s \neq s'$  in  $S$ , s.t.  $m_{s,s'}$  is finite and odd. Then 1.9: Let  $s = (s_1 s_2 \dots s_p)$ ,  $s' = (s'_1 s'_2 \dots s'_q)$  in  $X$ , be such that  $s_1 s_2 \dots s_p = s'_1 s'_2 \dots s'_q = w \in W$ . Then  $s \sim s'$ .

Let  $A = \mathbb{Z}\langle V \rangle$  where  $V$  is an indeterminate. For  $s \in S$ , we set  $V_s = V^{L(s)} \in A$ .

(1.8) Defn.  $X = (s_1, s_2, \dots, s_p)$ , reduced. We regard  $X$  as the vertices of a graph in which  $(s_1, s_2, \dots, s_p)$ ,  $(s'_1, s'_2, \dots, s'_q)$  are joined if one is obtained from the other by replacing in successive entries of form  $s_i, s'_i, s_i$  by the  $m_{s_i, s'_i}$  entries  $s'_i, s_i, s'_i, \dots$ ,  $(L(s), L(s') < \infty)$ .  $(s_1, s_2, \dots, s_p) \sim (s'_1, s'_2, \dots, s'_q)$ .

3.2. Let  $H$  be the  $A$ -algebra defined by the generators  $T_s (s \in S)$  and the relation

- (a)  $(T_s - V_s)(T_s + V_s) = 0$  for  $s \in S$
- (b)  $T_s T_{s'} \dots = T_{s'} T_s T_s \dots$   $\forall s \neq s'$  in  $S$

$H$  is called the Iwahori-Hecke algebra. We have  $1_H = 1$  the unit element of  $H$ .

Defn.  $T_w \in H$ ,  $T_w = T_{s_1} T_{s_2} \dots T_{s_p}$ , where  $w = s_1 s_2 \dots s_p$  reduced expression.

By (b) and (1.9),  $T_w$  is independent of the choice reduced expression.

From the definition,  $\forall s \in S, w \in W$   
 $T_s T_w = T_{sw}$  if  $L(sw) = L(w) + 1$   
 $T_s T_w = T_w + (V_s - V_s^{-1}) T_w$  if  $L(sw) = L(w) - 1$

$$T_w = T_s w = T_s T_w$$

$$T_s T_w = T_s T_s T_w \stackrel{(1.10)}{=} (V_s T_s - V_s^{-1} T_s + 1) T_w$$

$$= T_w + (V_s - V_s^{-1}) T_s T_w$$

$$= T_w + (V_s - V_s^{-1}) T_w$$

In particular,  $A$ -submodule of  $H$  generated by  $\{T_w; w \in W\}$  is a left ideal of  $H$ .

It contains  $1 = T_1$ , hence it's the whole of  $H$ .

Thus  $\{T_w; w \in W\}$  generates the  $A$ -module  $H$ .

3.3. Prop.  $\{T_w; w \in W\}$  is an  $A$ -basis of  $H$ .

Pf: the free  $A$ -module  $E$  with basis  $(e_w)_{w \in W}$

For any  $s \in S$ , define  $A$ -linear map  $P_s, Q_s: E \rightarrow E$

$$P_s(e_w) = e_{sw} \text{ if } L(sw) = L(w) + 1$$

$$P_s(e_w) = e_w + (V_s - V_s^{-1}) e_w \text{ if } L(sw) = L(w) - 1$$

$$Q_s(e_w) = e_{ws} \text{ if } L(ws) = L(w) + 1$$

$$Q_s(e_w) = e_w + (V_s - V_s^{-1}) e_w \text{ if } L(ws) = L(w) - 1$$

Assume: (a)  $P_s Q_t = Q_t P_s$  for  $\forall s, t$  in  $S$

Let  $\mathcal{A}$  be the  $A$ -subalgebra with 1 of  $\text{End}(E)$  generated by  $\{P_s, Q_s; s \in S\}$

the map  $\mathcal{A} \rightarrow E$   
 $\pi: P_s \mapsto P_s, Q_s \mapsto Q_s$

surjective: If  $w = s_1 s_2 \dots s_p$  is reduced, then  $e_w = P_{s_1} \dots P_{s_p} e_1$

injective: Assume  $\pi \in \mathcal{A}$  and  $\pi(e_1) = 0$ . Let  $\pi' = Q_{s_1} \dots Q_{s_1}$ , by (a)  $\pi \pi' = \pi' \pi$

$\therefore 0 = \pi' \pi(e_1) \stackrel{\text{by (a)}}{=} \pi \pi'(e_1) = \pi(Q_{s_1} \dots Q_{s_1}(e_1)) = \pi(e_w)$

$\therefore w$  is arbitrary,  $\therefore \pi = 0$

$\therefore$  an isomorphism of  $A$ -module.

Using the isomorphism  $\dots$  transport the algebra structure of  $\mathcal{A}$  to any algebra structure of  $E$  with the unit  $e_1$ .

$$P_s(e_1) \pi(e_1) = P_s(\pi(e_1)), \forall s \in S, w \in \mathcal{A}$$

$$\therefore e_s e_w = P_s(e_w), \forall w \in W, s \in S$$

$$\therefore (b) e_s e_w = e_{sw}, \text{ if } L(sw) = L(w) + 1$$

$$(c) e_s e_w = e_w + (V_s - V_s^{-1}) e_w, \text{ if } L(sw) = L(w) - 1$$

From (b) if  $w = s_1 s_2 \dots s_p$  then  $e_w = e_{s_1} e_{s_2} \dots e_{s_p}$

From (c)  $(w = s) e_s^2 = 1 + (V_s - V_s^{-1}) e_s, \forall s \in S$

$(e_s - V_s)(e_s + V_s^{-1}) = 0$

7 We see there is a unique algebra homomorphism  $H \rightarrow E$  preserving 1, s.t.  $T_s \mapsto e_s$  for all  $s \in S$

Assume  $a_w \in A (w \in W)$  are zero for all but finitely many  $w$  and that  $\sum_w a_w T_w = 0$  in  $H$

Applying  $H \rightarrow E, \sum_w a_w e_w = 0$

$\therefore (e_w)$  is a basis of  $E, \Rightarrow a_w = 0$

Thus  $\{T_w; w \in W\}$  is an  $A$ -basis of  $H$

Proof of (a):

Case 1.  $swt, sw, wt, w$  have lengths  $l+2, l+1, l+1, l$

Then  $P_s P_t(e_w) = P_t P_s(e_w) = e_{swt}$

Case 2.  $w, sw, wt, swt: l+2, l+1, l+1, l$

$$P_s Q_t(e_w) = Q_t P_s(e_w)$$

$$= e_{swt} + (V_s - V_s^{-1}) e_{sw} + (V_s - V_s^{-1}) e_{wt}$$

$$+ (V_s - V_s^{-1}) (V_s - V_s^{-1}) e_w$$

3:  $l+2, l+1, l+1, l$

4:  $sw, wt$

5:  $swt, \dots, sw, wt, \dots, wt, \dots, w, w$

6:  $\dots$

Chapter 4. The Bar Operator

4.1. For  $s \in S$ , the element  $T_s \in H$  is invertible,  $T_s^{-1} = T_s - (V_s - V_s^{-1})$

$$T_s T_s^{-1} = T_s (T_s - (V_s - V_s^{-1})) = T_s + (V_s - V_s^{-1}) T_s - T_s (V_s - V_s^{-1})$$

$$= T_s$$

$\therefore T_w$  is invertible for each  $w \in W$  if  $w = s_1 s_2 \dots s_p$  is reduced then  $T_w^{-1} = T_{s_p}^{-1} \dots T_{s_1}^{-1}$

let  $\bar{\cdot}: A \rightarrow A$  be the ring involution which takes  $V^n$  to  $V^{-n}, \forall n \in \mathbb{Z}$

4.2 LEMMA (a) There is a unique ring homomorphism  $\bar{\cdot}: A \rightarrow A, \bar{T}_s = T_s^{-1}, \forall s \in S$

(b) This homomorphism is involutive it takes  $T_w$  to  $\bar{T}_w$   $\forall w \in W$

Pf: (a)  $(T_s^{-1} - V_s^{-1})(T_s + V_s) = 0, \forall s \in S$

(b) let  $s \in S, \bar{T}_s \bar{T}_s = 1, \bar{T}_s = \bar{\bar{T}_s} = T_s$

4.3. For any  $w \in W, \bar{T}_w = \sum_{y \in W} \bar{r}_{yw} \bar{T}_y$   $r_{yw} \in A$ , are zero for all but finitely many  $y$

4.4. Lemma Let  $w \in W$  and  $s \in S, w > sw$  For  $y \in W, r_{yw} = T_s r_{y,sw}$  if  $sy > y$

Pf:  $\bar{T}_w = T_s^{-1} \bar{T}_{sw} = (T_s - (V_s - V_s^{-1})) \sum_{y \in W} r_{y,sw} \bar{T}_y$

$$= \sum_{y \in W} r_{y,sw} \bar{T}_y - \sum_{y \in W} (V_s - V_s^{-1}) r_{y,sw} \bar{T}_y + \sum_{y \in W} (V_s - V_s^{-1}) r_{y,s} \bar{T}_y$$

$$= \sum_{y \in W} r_{y,sw} \bar{T}_y - \sum_{y \in W} (V_s - V_s^{-1}) r_{y,sw} \bar{T}_y \quad \square$$

4.5. Lemma: For any  $y, w, \bar{r}_{yw} = \text{sgn}(yw) r_{yw}$

Pf:

mayl groupoid 乘法为0  $\rightarrow$  Hecke algebra