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Chapter 1 Coxeter Group

Defn 1.1 Let S be a finite set and let $(m_{s,s'})_{(s,s')\in S\times S}$ be a matrix with entries in $\mathbb{N}\cup\{\infty\}$, s.t. $m_{s,s}=1$ for all s and $m_{s,s'}=m_{s's}\geq 2$.
$(m_{s,s'})$ is called coxeter matrix

$\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$

Defn 1.2 In the case, where $m_{s,s'}\in\{2,3,4,6,\infty\}$ for all $s\neq s'$, the matrix $(m_{s,s'})_{(s,s')\in S\times S}$ is completely described by a graph (the Coxeter graph) with set of vertices in bijection with S where the vertices corresp to $s\neq s'$ are joined by
- an edge, if $m_{s,s'}=3$
- double edge, if $m_{s,s'}=4$
- triple edge, if $m_{s,s'}=6$
- quadruple edge, if $m_{s,s'}=\infty$

$\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}$

$\begin{pmatrix}1 & 4\\ 4 & 1\end{pmatrix}$

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$ss's=s'ss'$

$ss'ss'ss'=1$

Defn 1.3 Let W be a group generated by S,
$(ss')^{m_{s,s'}}=1$, $m_{s,s'}<\infty$, $s^2=1$

W is called a Coxeter group

Chap 3. The Algebra H

3.1. Let W,S be a Coxeter group.
A map $L: W\to\mathbb{Z}$ is said to be a weight function for W if $L(ww')=L(w)+L(w')$, $\forall\, w,w'\in W$, s.t. $l(ww')=l(w)+l(w')$

Defn: Assume that a weight function $L: W\to\mathbb{Z}$ is fixed, ... W,L is a weighted Coxeter group.

For ex. ..., $L=l$, split case
Note: L is determined by its value on S which are subject only to the condition.
? $L(s)=L(s')$ $\forall\, s\neq s'$ in S, s.t. $m_{s,s'}$ is finite and odd

$w=s_1s_2\cdots s_n$ $=s_1'\cdots s_n'$, reduced
$L(s_1s_2\cdots s_n)$
$=L(s_1)+L(s_2\cdots s_n)=L(s_1)+L(s_2)+L(s_3\cdots s_n)$
$=\cdots=L(s_1)+L(s_2)\cdots L(s_n)$

$s^2=1$, $(ss')^{m_{s,s'}}=1$ $\Rightarrow L(s)=L(s')$
$(s's)^2=(ss')^{-2}$
$(ss')=$
$(ss's)^m=1$
$L(s)$
$s's$ $s,s'=1$

$(ss')^{m_{s,s'}}=1$, 打开, $L(s)=L(s')$

$L(w)=L(w^{-1})$

Thm 1.9: Let $s=(s_1s_2\cdots s_q)$, $s'=(s_1's_2'\cdots s_q')$ in X, be such that $s_1s_2\cdots s_q=s_1's_2'\cdots s_q'=w\in$
Then $s\sim s'$

Let $A=\mathbb{Z}[v,v^{-1}]$ where v is an indeterminate,
For $s\in S$ we set $v_s=v^{L(s)}\in A$, v_s?

(1.8) Defn. $X=(s_1,s_2,\cdots s_q)$, reduced
We regard X as the vertices of a graph in which $(s_1,s_2,\cdots s_q)$, $(s_1',s_2',\cdots s_q')$ are joined if one is obtained from the other by replacing m consecutive entries of form $s,s',s,s'\cdots$ by the m entries $s',s,s',s\cdots$, $(s\neq s')$, $(m=m_{s,s'}<\infty)$
$(s_1,s_2,\cdots s_q)\sim(s_1',s_2',\cdots s_q')$

m? (ss')
$(s's)$
只差 $(ss')^m$

3.2. Let H be the A-algebra defined by the generators T_s $(s\in S)$ and the relation
(a) $(T_s-v_s T_1)(T_s+v_s^{-1}T_1)=0$ for $s\in S$
(b) $\underbrace{T_sT_{s'}T_s\cdots}_{m_{s,s'}}=\underbrace{T_{s'}T_sT_{s'}\cdots}_{m_{s,s'}}$ $\forall\, s\neq s'$ in S

H is called the Iwahori–Hecke algebra
We have $T_1=1$, the unit element of H

Defn. $T_w\in H$, $T_w=T_{s_1}T_{s_2}\cdots T_{s_q}$, where $w=s_1s_2\cdots s_q$ reduced expression

By (b) and (1.9), T_w is independent of the choice reduced expression.

From the definition, $\forall s\in S$, $w\in W$
$T_sT_w=T_{sw}$, if $l(sw)=l(w)+1$
$T_sT_w=T_{sw}+(v_s-v_s^{-1})T_w$ if $l(sw)=l(w)-1$

$T_w=T_{ssw}=T_sT_{sw}$
$T_sT_w=T_sT_sT_{sw}\xlongequal{3.2(a)}(v_sT_s-v_s^{-1}T_s+1)T_{sw}$
$=T_{sw}+(v_s-v_s^{-1})T_sT_{sw}$
$=T_{sw}+(v_s-v_s^{-1})T_w$

In particular, A-submodule of H generated by $\{T_w; w\in W\}$ is a left ideal of H
It contains $1=T_1$, hence it's the whole of H
Thus $\{T_w; w\in W\}$ generates the A-module H

3.3. Prop. $\{T_w; w\in W\}$ is an A-basis of H

Pf: the free A-module $\mathcal{E}$ with basis $(e_w)_{w\in W}$
For any $s\in S$, define A-linear map $P_s, Q_s:\mathcal{E}\to\mathcal{E}$
$P_s(e_w)=e_{sw}$ if $l(sw)=l(w)+1$
$P_s(e_w)=e_{sw}+(v_s-v_s^{-1})e_w$, if $l(sw)=l(w)-1$
$Q_s(e_w)=e_{ws}$, if $l(ws)=l(w)+1$
$Q_s(e_w)=e_{ws}+(v_s-v_s^{-1})e_w$, if $l(ws)=l(w)-1$

Assume: (a) $P_sQ_t=Q_tP_s$ for $\forall\, s,t$ in S

Let $\mathfrak{U}$ be the A-subalgebra with 1 of $\mathrm{End}(\mathcal{E})$ generated by $\{P_s; s\in S\}$

the map $\mathfrak{U}\to\mathcal{E}$
$\pi\mapsto\pi(e_1)$

surjective: If $w=s_1s_2\cdots s_q$ is reduced, the $e_w=P_{s_1}\cdots P_{s_q}e_1$

injective: Assume $\pi\in\mathfrak{U}$ and $\pi(e_1)=0$
Let $\pi'=Q_{s_q}\cdots Q_{s_1}$, By (a) $\pi\pi'=\pi'\pi$
$\therefore 0=\pi'\pi(e_1)\xlongequal{\text{by (a)}}\pi\pi'(e_1)=\pi(Q_{s_q}\cdots Q_{s_1}(e_1))=\pi(e_w)$
$\because w$ is arbitrary, $\therefore \pi=0$
$\therefore$ an isomorphism of A-module.
Using the isomorphism .. transport the algebra structure of $\mathfrak{U}$ to any algebra structure of $\mathcal{E}$ with the unit e_1

$P_s(e_1)\pi(e_1)=P_s(\pi(e_1))$, $\forall s\in S$, $w\in\mathfrak{U}$
$\therefore e_se_w=P_s(e_w)$ $\forall\, w\in W$, $s\in S$
$\therefore$ (b) $e_se_w=e_{sw}$, if $l(sw)=l(w)+1$
(c) $e_s(e_w)=e_{sw}+(v_s-v_s^{-1})e_w$, if $l(sw)=l(w)-1$

From (b) if $w=s_1s_2\cdots s_q$ ~~then~~ $e_w=e_{s_1}e_{s_2}\cdots e_{s_q}$
if $s\neq s'\in S$, $m=m_{s,s'}<\infty$
$e_se_{s'}e_s\cdots=e_{s'}e_se_{s'}$

From (c) $(w=s)$ $e_s^2=1+(v_s-v_s^{-1})e_s$, $\forall s\in S$
$(e_s-v_s)(e_s+v_s^{-1})=0$

? We see the there is a unique algebra homomorphism $H\to\mathcal{E}$ preserving 1, s.t. $T_s\mapsto e_s$ for all $s\in S$
Assume $a_w\in A$ $(w\in W)$ are zero for all but finitely many w and that $\sum_w a_wT_w=0$ in H
Applying $H\to\mathcal{E}$, $\sum_w a_e e_w=0$
$\because (e_w)$ is a basis of $\mathcal{E}$, $\Rightarrow a_w=0$
Thus $\{T_w: w\in W\}$ is an A-basis of H

Proof of (a):
Case 1. swt, sw, wt, w have lengths $q+2, q+1, q+1, q$
Then $P_sP_t(e_w)=Q_tP_s(e_w)=e_{swt}$
Case 2. w, sw, wt, swt: $q+2, q+1, q+1, q$
$P_sQ_t(e_w)=Q_tP_s(e_w)$
$=e_{swt}+(v_t-v_t^{-1})e_{sw}+(v_s-v_s^{-1})e_{wt}$
$+(v_t-v_t^{-1})(v_s-v_s^{-1})e_w$
3: $q+2, q+1, q+1, q$
4: sw wt
5: $swt, \cdots sw$ $q+1, q+1, q, q$
6: $\square$

Chapter 4. The Bar Operator

4.1. For $s\in S$, the element $T_s\in H$ is invertible,
$T_s^{-1}=T_s-(v_s-v_s^{-1})$

$T_sT_s^{-1}=T_s(T_s-(v_s-v_s^{-1}))=T_1+(v_s-v_s^{-1})T_s-T_s(v_s-v_s^{-1})$
$=T_1$
$\therefore T_w$ is invertible for each $w\in W$
if $w=s_1s_2\cdots s_q$ is reduced
then $T_w^{-1}=T_{s_q}^{-1}\cdots T_{s_2}^{-1}T_{s_1}^{-1}$

let $\bar{\ }: A\to A$ be the ring involution which takes v^n to v^{-n}, $\forall n\in\mathbb{Z}$

4.2. LEMMA (a) There is a unique ring homomorphism $\bar{\ }: A\to A$, $\overline{T_s}=T_s^{-1}$ $\forall$ s
(b) This homomorphism is involutive it takes T_w to $T_{w^{-1}}^{-1}$ $\forall\, w\in W$

Pf: (a) $(T_s^{-1}-v_s^{-1})(T_s^{-1}+v_s)=0$, $\forall s\in S$
$T_s^{-1}T_{s'}^{-1}T_s^{-1}\cdots=T_{s'}^{-1}T_s^{-1}T_{s'}^{-1}\cdots$ $\square$
(b) let $s\in S$ $T_s\overline{T_s}=1$, $\overline{T_s}\,\overline{\overline{T_s}}=$
$\overline{T_s}T_s=1$ $\therefore \overline{\overline{T_s}}=T_s$
the square of $\bar{\ }$ is 1

生成元已定义

$(T_s-v_s)(T_s+v_s^{-1})=0$
$(T_sT_s^{-1}-v_sT_s^{-1})(1+v_s^{-1}T_s^{-1})=0$
$(v_s^{-1}-T_s^{-1})(v_s+T_s^{-1})=0$
$T_sT_{s'}T_s\cdots=T_{s'}T_sT_{s'}\cdots$
$T_{s'}T_s\cdots=T_s^{-1}T_{s'}T_s$

4.3. For any $w\in W$, $\overline{T_w}=\sum_{y\in W}\overline{r_{y,w}}T_y$
$r_{y,w}\in A$, are zero for all by finitely many y

$T_sT_{sw}=T_{ssw}+(v_s-v_s^{-1})T_{sw}$

4.4. Lemma Let $w\in W$ and $s\in S$, $w>sw$
$w=ssw$
For $y\in W$, $r_{y,w}=r_{sy,sw}$, if sy
$\overline{T_{ssw}}=\overline{T_sT_{sw}}=\overline{T_s}\,\overline{T_{sw}}$ $r_{y,w}=s_{y,sw}+(v_s-v_s^{-1})r_{y,sw}$, if $sy>y$
Pf. $\overline{T_w}=T_s^{-1}\overline{T_{sw}}=(T_s-(v_s-v_s^{-1}))\sum_y\overline{r_{y,sw}}T_y$
$=\sum_y\overline{r_{y,sw}}T_{sy}-\sum_y(v_s-v_s^{-1})\overline{r_{y,sw}}T_y+\sum_{y;sy<y}(v_s-v_s^{-1})\overline{r_{y,s}}$
$=\sum_y\overline{r_{sy,sw}}T_y-\sum_{s;sy>y}(v_s-v_s^{-1})\overline{r_{y,sw}}T_y$ $\square$

4.5. Lemma: For any y,w, $\overline{r_{y,w}}=\mathrm{sgn}(yw)r_{y,w}$
Pf:

Weyl groupoid $\xrightarrow{\text{乘法为0}}$ Hecke algebra